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State Estimation with Multi-packet Transmission Over the Wireless Network

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Abstract: In wireless control, the measurement of a system may be divided into multiple partial observations for transmission purpose, known as "multi-packet transmission". The state estimation for the discrete linear time-invariant system with the multi-packet transmission is considered. We focus on what the filter has received during each time step. For the randomness of the received packets, the classical Kalman filter is revised to the time-varying Kalman filter. With the concept of combination and the supposed delivery rate for each packet, we analyze the stability of the filter in the sense of expectation. To deal with the combination explosion problem, alternative conditions on delivery rates is provided.

Key Words: Kalman filtering, networked control systems, multi-packet transmission, stability

1 Introduction

The fast development of the wireless communication technology enables the control links to be closed by various forms of wireless communication networks, forming the so-called Wireless Networked Control Systems (WNCSs) [1–3]. wireless communication makes it possible that the control systems are more easily implemented and deployed, thus yielding many existing and future applications, including, for example, factory automation[4], smart grid [5], smart home[6], multi-robot control in logistics and climate control in agricultural greenhouse [7, 8], and so on. These benefits, however, are not obtained at no expenses. In fact, wireless control deteriorates the control performance comparing with the traditional wired control, because of the more limited bandwidth and even worse communication quality.

The imperfect communication in WNCSs makes it a challenging task to estimate the system state, as more uncertainties are introduced to the system. One particular case is where the measurement of a system may be divided into multiple partial observations for transmission purpose, that is,"multi-packet transmission". Since each of the partial observation is encoded into independent data packet and sent separately, they may experience different delays or dropouts. This makes the state estimation even challenging. A popular reason why this situation may occur is due to the multiple sensors used in the system, making it difficult to pack all the sensing data together to transmit. Clearly this situation can be even worse in WNCSs due to its usually flexible and complex system structure [9].

The problem of Kalman filtering with measurements dropping has already been studied intensively [10–15]. These papers focus on the one packet case, where the measurements are either received or lost as a whole. The stability of the Kalman filter is often analyzed by modelling the transmission as the Bernoulli process or Markov process. Kalman filtering with transmission by two packets is studied in [16]. However, only a few articles have studied the multi-packet transmission case [17, 18]. In [17], the authors show that the stability of the estimator depends on the rate of packet loss patterns, resulting in the column rank deficiency of an N-step regression matrix, while under some assumptions. With the convergence property of Kalman filter in gathered and dispersed measurement case, the bounds on critical probability of receiving packets are computed by solving convex minimization problems in [18], while the computational burden is heavy when the number of packets is large.

In this work, motivated by practical applications, the results in [10] are extended to the multi-packet transmission case. The linear matrix inequality (LMI) conditions on delivery rates to bound the expected error covariance matrix is provided. To deal with the dimension explosion problem of the original LMI condition, alternative LMI conditions whose dimension increases along with the growth of packet number are also provided.

The remainder of the paper is organized as follows. In Section 2, we introduce the system of interest, the time-varying Kalman filter and the combination of partial observations. In Section 3, we analyze the statistic properties of the expected error covariance matrix and present the alternative LMI condition on the delivery rates. Finally, we show some simulation results in Section 4 and conclude the paper in Section 5.

Notations: $\mathbb{E}[x]$ denotes the expectation of x and $\mathbb{E}[x|y]$ is the conditional expectation. $x|_y$ denotes the result of x when y is given. $Pr\{\}$ means probability. The transpose of matrix A is denoted as A' and the diagonal matrix with the main diagonal elements $l_1, l_2, \ldots, l_m$ is denoted as $diag\{l_1, l_2, \ldots, l_m\}$. The counterparts for elements of symmetric matrix are replaced with $*$. Matrix $P > 0 (\geq 0)$ means the matrix P is positive definite (positive semi-definite).

2 PROBLEM STATEMENT AND PRELIMINARIES

This paper considers the Kalman filtering for a linear-invariant system controlled over the wireless network, where the sensors access the wireless network through r interfaces, as depicted in Fig.1. At single time step, r packets are transmitted over the wireless network with each packet occupying

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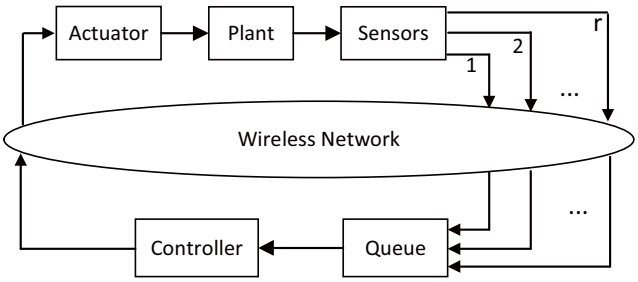

Fig. 1: Diagram of wireless control with multi-packet transmission.

one interface, which can be a wireless channel or a time slot. During each time step, the correctly received packets would be stored in a queue as the current received data (CRD), which will be taken by the controller (filter) as a whole bofore the next time step comes. In this case, the filter may receive complete, partial or even empty measurement according to the CRD. Thus, our problems include how to address the CRD and how the filter iterates under this situation. Mostly and importantly, we are interested in how the delivery rates of packets affect the iteration of the filter. Before tackling these problems, we briefly introduce the classical Kalman filter.

Consider the following discrete linear-invariant system,

$$\begin{aligned} x_{k+1} &= Ax_k + \omega_k \\ y_k &= Cx_k + \nu_k \end{aligned} \tag{1}$$

where $A \in R^{n\times n}$ is the system matrix and $C \in R^{m\times n}$ is the measurement matrix. $x_k \in R^n$ is the system state, $y_k \in R^m$ is the measurement output, ω_k denotes the system disturbance, υ_k denotes the measurement noise, and the subscript k is the time index. ω_k, υ_k and the initial state x_0 are uncorrelated Gaussian random vectors with appropriate dimensions. ω_k and υ_k have zero means and covariance matrices $Q \geq 0$ and $R > 0$, respectively. Denotes $y_0^k = \{y_0, y_1, \ldots, y_k\}$ as the measurements history, and define

$$\begin{aligned} \hat{x}_{k|k} &\triangleq \mathbb{E}[x_k|y_0^k] \\ P_{k|k} &\triangleq \mathbb{E}[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})'|y_0^k] \\ \hat{x}_{k+1|k} &\triangleq \mathbb{E}[x_{k+1}|y_0^k] \\ P_{k+1|k} &\triangleq \mathbb{E}[(x_{k+1} - \hat{x}_{k+1|k})(x_{k+1} - \hat{x}_{k+1|t})'|y_0^k]. \end{aligned}$$

The classical Kalman filter can be designed as follows, with the time update being

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} \\ P_{k+1|k} &= AP_{k|k}A' + Q, \end{aligned} \tag{2}$$

and the measurement update being

$$\begin{aligned} \hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + K_{k+1}(y_{k+1} - C\hat{x}_{k+1|k}) \\ K_{k+1} &= P_{k+1|k}C'(CP_{k+1|k}C' + R)^{-1} \\ P_{k+1|k+1} &= P_{k+1|k} - K_{k+1}CP_{k+1|k}. \end{aligned} \tag{3}$$

Combining (2)(3), we obtain one-step prediction equation

Table 1: Combinations of partial observations

Number	combinations				$\mathcal{C}_i$, $\mathcal{R}_i$	λ_i
	σ_r	...	σ_2	σ_1		
0	0	0	0	0	$\mathcal{C}_0$, $\mathcal{R}_0$	λ_0
1	0	0	0	1	$\mathcal{C}_1$, $\mathcal{R}_1$	λ_1
2	0	0	1	0	$\mathcal{C}_2$, $\mathcal{R}_2$	λ_2
3	0	0	1	1	$\mathcal{C}_3$, $\mathcal{R}_3$	λ_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	1	1	1	1	$\mathcal{C}_N$, $\mathcal{R}_N$	λ_N

and the algebraic riccati equation

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + K_k(y_k - C\hat{x}_{k|k-1}) \tag{4}$$

$$\begin{aligned} P_{k+2|k+1} &= AP_{k+1|k}A' + Q \\ &\quad -AP_{k+1|k}C'(CP_{k+1|k}C' + R)^{-1}CP_{k+1|k}A'. \end{aligned} \tag{5}$$

When (A, Q) is controllable and (A, C) is observable, the classical Kalman filter is stable, i.e. the equation (5) converges to a positive semidefinite matrix [19].

From the above statement, one can observe that the classical Kalman filter can not be directly applied to multi-packet transmission case, which, therefore, should be revised to adapt to the random CRD. We now formulate the CRD as follows.

By expanding the measurement equation (1) according to its partial observations contained in the r packets, we get

$$\begin{bmatrix} y_{1,k} \\ y_{2,k} \\ \vdots \\ y_{r,k} \end{bmatrix} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_r \end{bmatrix} x_k + \begin{bmatrix} \nu_{1,k} \\ \nu_{2,k} \\ \vdots \\ \nu_{r,k} \end{bmatrix} \tag{6}$$

where $y_{i,k} \in R^{s_i}$ is the patial observation contained in the ith packet whose measurement noise is $\upsilon_{i,k}$ ($\sum_{i=1}^{r} s_i = m$). It is assumed that $\upsilon_{i,k}$ are Gaussian random vectors with zero mean and covariance matrices $R_i > 0$ and $R = diag\{R_1, R_2, \ldots, R_r\}$ [20].

The CRD is a set of data packets which are temporarily stored in the queue. Hence, the randomness of the CRD is caused by two aspects,

1) the content of CRD is random, which can be arbitrary combination of the partial observations and even empty.

2) the sequence of CRD is random. This is because the packets randomly arrive at the queue and are stored in random sequence. This considers the wireless network where the packets contend to access the shared wireless channel and transmitted in random time slots, such as the IEEE 802.15.4 protocol.

We use a set of binary variables to describe the content of CRD for each time step

$$\sigma_i = \begin{cases} 1 & \text{the } i\text{th partial observation is included in CRD} \\ 0 & \text{the } i\text{th partial observation is lost} \end{cases}$$

and we assume $Pr\{\sigma_i = 1\} = p_i$, which represents the delivery rate of the partial observation i. Now we let the content of CRD corresponds to a combination of these binary variables, as shown in Tab. 1. Denote $\hat{y}_k$ as the CRD of time

step k, similar to (6), we have $\hat{y}_k \equiv \hat{C}_k x_k + \hat{\nu}_k$. Denote $\hat{R}_k$ as covariance matrix of $\hat{\nu}_k$. The CRD of time step k now can also be represented as the set $\{\hat{y}_k, \hat{C}_k, \hat{R}_k\}$.

Example 1. *When* $\hat{y}_k = \begin{bmatrix} y_{3,k} \\ y_{1,k} \end{bmatrix}$ *is stored in the queue, then we have* $\hat{C}_k = \begin{bmatrix} C_3 \\ C_1 \end{bmatrix}$ *and* $\hat{R}_k = \begin{bmatrix} R_3 & 0 \\ 0 & R_1 \end{bmatrix}$.

With the input of CRD, the classical Kalman filter is revised as the time-varying Kalman filter [20],

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + K_k(\hat{y}_k - \hat{C}_k\hat{x}_{k|k-1}) \tag{7}$$

$$P_{k+1|k} = AP_{k|k-1}A' + Q - K_k\hat{C}_kP_{k|k-1}A' \tag{8}$$

$$K_k = AP_{k|k-1}\hat{C}_k'(\hat{C}_kP_{k|k-1}\hat{C}_k' + \hat{R}_k)^{-1}$$

Remark 1. *The one-step predicton of the state is often used because the estimator has to wait for a time step to receive all the packets with diffrent transmission delays [21].*

For symplicity, $P_{k|k-1}$, $P_{k+1|k}$ are replaced with P_k, P_{k+1} respectively. Since the $\hat{C}_k$ and $\hat{R}_k$ are random, the analysis of the filter's iteration is pretty complex. We notice that the iteration is only dependent on the content of CRD and independent of the sequence, as stated in the following Lemma.

Lemma 2.1. *Suppose* $\hat{y}_{k,1}$ *and* $\hat{y}_{k,2}$ *are two CRDs with the same content but different sequence, then,*

$$\hat{x}_{k+1|k}|_{\hat{y}_{k,1},P_k,\hat{x}_{k|k-1}} = \hat{x}_{k+1|k}|_{\hat{y}_{k,2},P_k,\hat{x}_{k|k-1}} \tag{9}$$

$$P_{k+1}|_{\hat{y}_{k,1},P_k} = P_{k+1}|_{\hat{y}_{k,2},P_k} \tag{10}$$

Proof. Note that only the term $\hat{C}_k'(\hat{C}_kP_k\hat{C}_k' + \hat{R}_k)^{-1}\hat{C}_k$ need to be considered. Suppose $\{\hat{y}_{k,1}, \hat{C}_{k,1}, \hat{R}_{k,1}\}$ and $\{\hat{y}_{k,2}, \hat{C}_{k,2}, \hat{R}_{k,2}\}$ are two $CRDs$ with the same partial observations but different queue sequence. Then we have $\hat{y}_{k,1} = T\hat{y}_{k,2}$, where $T \equiv T_lT_{l-1}\dots T_2T_1$, $(l \leq r)$ and T_i, $i \in \{l, l-1, \dots, 2, 1\}$, is row operation matrix which exchanges two partial observations in $\hat{y}_{k,2}$. Then,

$$\begin{aligned}
&\hat{C}_{k,1}'(\hat{C}_{k,1}P_k\hat{C}_{k,1}' + \hat{R}_{k,1})^{-1}\hat{C}_{k,1} \\
=&(T\hat{C}_{k,2})'[T\hat{C}_{k,2}P_k(T\hat{C}_{k,2})' + T\hat{R}_{k,2}T']^{-1}T\hat{C}_{k,2} \\
=&\hat{C}_{k,2}'T'[T(\hat{C}_{k,2}P_k\hat{C}_{k,2}' + \hat{R}_{k,2})T']^{-1}T\hat{C}_{k,2} \\
=&\hat{C}_{k,2}'(\hat{C}_{k,2}P_k\hat{C}_{k,2}' + \hat{R}_{k,2})^{-1}\hat{C}_{k,2}
\end{aligned}$$

this completes the proof. □

The time-varying Kalman filter is actually driven by a specific combination of partial observations in each step. We sum up all the combinations in Tab. 1, where $N = 2^r - 1$, $\mathcal{C}_i$ and $\mathcal{R}_i$ represent a set of matrice corresponding to the Combination i. λ_i is the probability when Combination i occurs and we have

$$\lambda_i = \prod_{j=1}^{r} \sigma_j p_j + (1-\sigma_j)(1-p_j) \tag{11}$$

given the Combination i and

$$\sum_{j=0}^{N} \lambda_j = \prod_{j=1}^{r} [p_j + (1-p_j)] = 1 \tag{12}$$

Remark 2. *By the eqution* (10)*, we can use* $\mathcal{C}_i$*,* $\mathcal{R}_i$ *to replace all* $\hat{C}_k$*,* $\hat{R}_k$ $(k = 1, 2, \dots)$ *in the filter when analyzing the iteration of the error covariance matrix, as long as* $\hat{C}_k$*,* $\hat{R}_k$ *correspond to combination* i*.*

3 STATISTICAL PROPERTIES AND CONVERGENCE CONDITION

In this section, we will state the statistical properties in the iteration of the expected error covariance matrix and sufficient condition to bound $\mathbb{E}[P_k]$. First, we present the similar formulations in [18]:

$$\begin{aligned}
g_{p_1p_2\dots p_r}(X) &= AXA' + Q \\
&- \sum_{i=1}^{N} \lambda_i AX\mathcal{C}_i'(\mathcal{C}_iX\mathcal{C}_i' + \mathcal{R}_i)^{-1}\mathcal{C}_iXA'
\end{aligned} \tag{13}$$

where $X \geq 0$ and

$$\mathbb{E}[P_{k+1}|P_k] = g_{p_1p_2\dots p_r}(P_k) \tag{14}$$

$$\mathbb{E}[P_{k+1}] = \mathbb{E}[g_{p_1p_2\dots p_r}(P_k)] \tag{15}$$

Lemma 3.1. *The function* $g_{p_1p_2\dots p_r}(X)$ *has the following properties.*

(a) If $X \leqslant Y$, $g_{p_1p_2\dots p_r}(X) \leqslant g_{p_1p_2\dots p_r}(Y)$.

(b) If $p_i^{(1)} \leq p_i^{(2)}$*, and* $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_r$ *are fixed. Then,* $g_{p_1\dots p_i^{(1)}\dots p_r}(X) \geq g_{p_1\dots p_i^{(2)}\dots p_r}(X)$.

(c) If X *is a random variable, then* $\lambda_0 A\mathbb{E}[X]A' + Q \leq \mathbb{E}[g_{p_1p_2\dots p_r}(X)] \leq g_{p_1p_2\dots p_r}(\mathbb{E}[X])$.

Proof. Here we only provide the proof of the term (b). The others are similar to the proofs in [10].

(b) We only study the case when $i = 1$ and the results hold when $i = 2, 3 \dots r$. First, λ_i is denoted as $\lambda_i^{(1)}, \lambda_i^{(2)}$ when p_1 equals $p_1^{(1)}, p_1^{(2)}$ respectively. We notice that $\lambda_i^{(2)}/(1-p_1^{(2)}) = \lambda_i^{(1)}/(1-p_1^{(1)}) = \lambda_{i+1}^{(2)}/p_1^{(2)} = \lambda_{i+1}^{(1)}/p_1^{(1)} \triangleq \lambda_{i,even}$ when i is even. Thus, $g_{p_1^{(2)}p_2\dots p_r}(X) - g_{p_1^{(1)}p_2\dots p_r}(X)$

$$\begin{aligned}
&= \textstyle\sum_{i=1}^{N}(\lambda_i^{(1)} - \lambda_i^{(2)})AX\mathcal{C}_i'(\mathcal{C}_iX\mathcal{C}_i' + \mathcal{R}_i)^{-1}\mathcal{C}_iXA' \\
&= \textstyle\sum_{n=0}^{\frac{N-1}{2}}(p_1^{(1)} - p_2^{(1)})\lambda_{2n,even}\times \\
&\quad [AX\mathcal{C}_{2n+1}'(\mathcal{C}_{2n+1}X\mathcal{C}_{2n+1}' + \mathcal{R}_{2n+1})^{-1}\mathcal{C}_{2n+1}XA' \\
&\quad - AX\mathcal{C}_{2n}'(\mathcal{C}_{2n}X\mathcal{C}_{2n}' + \mathcal{R}_{2n})^{-1}\mathcal{C}_{2n}XA'] \\
&= \textstyle\sum_{n=0}^{\frac{N-1}{2}}(p_1^{(1)} - p_1^{(2)})\lambda_{2n,even}AX\times \\
&\quad [\mathcal{C}_{2n+1}'(\mathcal{C}_{2n+1}X\mathcal{C}_{2n+1}' + \mathcal{R}_{2n+1})^{-1}\mathcal{C}_{2n+1} \\
&\quad - \mathcal{C}_{2n}'(\mathcal{C}_{2n}X\mathcal{C}_{2n}' + \mathcal{R}_{2n})^{-1}\mathcal{C}_{2n}]XA'
\end{aligned}$$

Next, we proof $\mathcal{C}_{2n+1}'(\mathcal{C}_{2n+1}X\mathcal{C}_{2n+1}' + \mathcal{R}_{2n+1})^{-1} \times \mathcal{C}_{2n+1} - \mathcal{C}_{2n}'(\mathcal{C}_{2n}X\mathcal{C}_{2n}' + \mathcal{R}_{2n})^{-1}\mathcal{C}_{2n} \triangleq \triangle\mathcal{C}_{(n)} \geq 0$. According to the lemma 2.1, we let $\mathcal{C}_{2n+1} = [C_1', \mathcal{C}_{2n}']'$ and $(\mathcal{C}_{2n+1}X\mathcal{C}_{2n+1}' + \mathcal{R}_{2n+1})^{-1} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, where $M_{12} = M_{21}'$, $M_{11} \in R^{s_1 \times s_1}$ and M_{22} are symmetric positive definite matrice, by using the inverse of a partitioned matrix [22], we have $(\mathcal{C}_{2n}X\mathcal{C}_{2n}' + \mathcal{R}_{2n})^{-1} = M_{22} - M_{21}M_{11}^{-1}M_{12}$. Since $\mathcal{C}_{2n}'(\mathcal{C}_{2n}X\mathcal{C}_{2n}' + \mathcal{R}_{2n})^{-1}\mathcal{C}_{2n}$

$$\begin{aligned}
&= \mathcal{C}_{2n+1}'\begin{bmatrix} 0 & 0 \\ 0 & (\mathcal{C}_{2n}X\mathcal{C}_{2n}' + \mathcal{R}_{2n})^{-1} \end{bmatrix}\mathcal{C}_{2n+1} \\
&= \mathcal{C}_{2n+1}'\begin{bmatrix} 0 & 0 \\ 0 & M_{22} - M_{21}M_{11}^{-1}M_{12} \end{bmatrix}\mathcal{C}_{2n+1}, \triangle\mathcal{C}_{(n)} =
\end{aligned}$$

$$\mathcal{C}'_{2n+1}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}-(M_{22}-M_{21}M_{11}^{-1}M_{12}) \end{bmatrix}\mathcal{C}_{2n+1}$$
$$=\mathcal{C}'_{2n+1}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{21}M_{11}^{-1}M_{12} \end{bmatrix}\mathcal{C}_{2n+1}.$$

Since $M_{11}>0$, $M_{21}M_{11}^{-1}M_{12}-M_{21}M_{11}^{-1}M_{12}=0$, we have $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{21}M_{11}^{-1}M_{12} \end{bmatrix}\geq 0$, [22]. If and only if $C_1=M_{11}^{-1}M_{12}\mathcal{C}_{2n}$ the equation holds.

□

Remark 3. *Based on the Lemma 3.1 (b), the critical rate for each packet can be found by constantly decreasing its delivery rate with the other rates fixed until the* $\lim_{k\to\infty}\mathbb{E}[P_k]$ *diverges.*

Combining the equation (15) *and Lemma 3.1 (c), we have*

$$\mathbb{E}[P_{k+1}]\leq g_{p_1p_2\dots p_r}(\mathbb{E}[P_k]) \tag{16}$$

Assume the sequence $V_{k+1}=g_{p_1p_2\dots p_r}(V_k)$, $V_0=\mathbb{E}[P_0]$, *then* $\mathbb{E}[P_1]\leq V_1$, *using Lemma 3.1 (a),* $\mathbb{E}[P_2]\leq V_2$. *Obviously we have* $\mathbb{E}[P_k]\leq V_k$ *for all* k. *So if we find the condition that bounds the sequence* $\{V_k\}$, *it also bounds the sequence* $\{\mathbb{E}[P_k]\}$.

According to the discussion in [16], the main results therein can be extended to the case where the measurement is sent with r packets. Thus, we directly present the following two lemmas without proofs.

Lemma 3.2. *Suppose* (A,Q) *is controllable, if* $\exists\bar{Z}_1,\bar{Z}_2,\dots,\bar{Z}_N$ *and* $0<\bar{Y}\leq I$ *such that* $\Psi_N(Y,Z_1,Z_2,\dots,Z_N)=$

$$\begin{bmatrix} Y & Y_N & Y_{N-1} & \dots & Y_1 & Y_0 \\ * & Y & 0 & \dots & 0 & 0 \\ * & * & Y & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & * & \dots & Y & 0 \\ * & * & * & \dots & * & Y \end{bmatrix}>0, \tag{17}$$

where $Y_i=\sqrt{\lambda_i}(YA+Z_i\mathcal{C}_i)$, $i=N,N-1,\dots,1$, $Y_0=\sqrt{\lambda_0}YA$.

Then, $\forall P_0\geq 0$, *the iteration* $V_{k+1}=g_{p_1p_2\dots p_r}(V_k)$ *is bounded.*

Lemma 3.3. *If the rate* $p_1,p_2,\dots,p_r$ *satisfies* $\Psi_N(Y,Z_1,Z_2,\dots,Z_N)>0$, *then the solution of* $\bar{V}=g_{p_1p_2\dots p_r}(\bar{V})$ *is given by*

$$\bar{V}=\arg\max_{V}\quad Trace(V),\quad \textit{subject to}\quad \Gamma(V)=$$
$$\begin{bmatrix} AVA'+Q-V & \mathcal{V}_N & \mathcal{V}_{N-1} & \dots & \mathcal{V}_1 \\ * & V_N & 0 & \dots & 0 \\ * & * & V_{N-1} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ * & * & * & \dots & V_1 \end{bmatrix}\geq 0, \tag{18}$$

where $V_i=\mathcal{C}_iV\mathcal{C}_i'+\mathcal{R}_i$, $\mathcal{V}_i=\sqrt{\lambda_i}AV\mathcal{C}_i'$, $i=N,N-1,\dots,1$.

One can easily observe that the dimension of matrice $\Gamma(V)$ and $\Psi_N(Y,Z_1,Z_2,\dots,Z_N)$ will exponentially increase along with the number of partial observations, which inevitably cause heavy burden for calculation. Inspired by the theory of information, we provide alternative LMI conditions with linear growth which, to a great extent, can replace the original condition of $\Psi_N(Y,Z_1,Z_2,\dots,Z_N)>0$. Before that we introduce the recursive equation for Fisher information matrix (FIM) [18].

$$\mathcal{J}_{k+1}=Q^{-1}$$
$$-Q^{-1}A(\mathcal{J}_k+A'Q^{-1}A+C'R^{-1}C)^{-1}AQ^{-1} \tag{19}$$

For optimality of the Kalman filter, the equation (5) and the recursive equation for FIM (19) converge to the same matrix, i.e. $\lim_{k\to\infty}P_k=\mathcal{J}_k^{-1}$, the Cramer-Rao lower bound (CRLB) [23]. Accordidng to the discussion in [18], the recursive equation for FIM in multi-packet transmission case can be revised as

$$\mathcal{J}_{k+1}^{mp}=Q^{-1}-Q^{-1}A\times$$
$$(\mathcal{J}_k^{mp}+A'Q^{-1}A+\sum_{i=1}^{N}\lambda_i\mathcal{C}_i'\mathcal{R}_i^{-1}\mathcal{C}_i)^{-1}AQ^{-1} \tag{20}$$

Since $\mathcal{R}_i$ are diagonal matrice with R_i on the pricipal diagonal, we also derive

$$\mathcal{J}_{k+1}^{mp}=Q^{-1}-Q^{-1}A\times$$
$$(\mathcal{J}_k^{mp}+A'Q^{-1}A+\sum_{i=1}^{r}p_iC_i'R_i^{-1}C_i)^{-1}AQ^{-1} \tag{21}$$

If $p_1=p_2=\dots=p_r=p_0$, we abtain

$$\mathcal{J}_{k+1}^{mp}=Q^{-1}-Q^{-1}A$$
$$(\mathcal{J}_k^{mp}+A'Q^{-1}A+p_0C'R^{-1}C)^{-1}AQ^{-1} \tag{22}$$

this is exactly the recursive equation for FIM in the single packet transmission case. The term $p_iC_i'R_i^{-1}C_i$ can be regarded as the measurement information of partial observation i and $\sum_{i=1}^{r}p_iC_i'R_i^{-1}C_i$ is the total measurement information of r partial observations. When the the delivery rates of the packets change, the total measurement information also changes. When the total measurement information is not enough to observe the underlying system states, even the optimal Kalman filter will not stable, i.e. the iteration of $P_{k+1}=g_{p_1p_2\dots p_r}(P_k)$ is unbounded and the LMI condition (17) can not be satisfied.

Based on the above intuition, we define

$$f_{p_1p_2\dots p_r}(X)=AXA'+Q$$
$$-\sum_{i=1}^{r}p_iAX\mathcal{C}_{2^{i-1}}'(\mathcal{C}_{2^{i-1}}X\mathcal{C}_{2^{i-1}}'+\mathcal{R}_{2^{i-1}})^{-1}\mathcal{C}_{2^{i-1}}XA'$$

Notice that the FIM recursive equation for iteration $P_{k+1}=f_{p_1p_2\dots p_r}(P_k)$ is exactly the equation of (21). That is to say, the two iterations of $P_{k+1}=g_{p_1p_2\dots p_r}(P_k)$ and $P_{k+1}=f_{p_1p_2\dots p_r}(P_k)$ are based on the same total measurement information. If the total measurement information is deficiency to observe the system state, the two iterations will both diverge. And because the two are both

Kalman filter like iterations, if $P_{k+1} = f_{p_1p_2\ldots p_r}(P_k)$ is bounded, to a great extent, we think $P_{k+1} = g_{p_1p_2\ldots p_r}(P_k)$ is also bounded. Similar LMI condition to bound $P_{k+1} = f_{p_1p_2\ldots p_r}(P_k)$ is present as follows

Theorem 1. *Suppose (A, Q) is controllable and $\sum_{i=1}^{r} p_i = p_{sum} < 1$. If $\exists \bar{Z}_1, \bar{Z}_2, \ldots, \bar{Z}_r$ and $0 < \bar{Y} \leq I$ such that $\Psi_r^1(Y, Z_1, Z_2, \ldots, Z_r) =$*

$$\begin{bmatrix} Y & Y_r & Y_{r-1} & \ldots & Y_1 & Y_0 \\ * & Y & 0 & \ldots & 0 & 0 \\ * & * & Y & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & * & \ldots & Y & 0 \\ * & * & * & \ldots & * & Y \end{bmatrix} > 0, \qquad (23)$$

where $Y_i = \sqrt{p_i}(YA + Z_i\mathcal{C}_{2^{i-1}})$, $i = r, r-1, \ldots, 1$, $Y_0 = \sqrt{1 - p_{sum}}YA$.

Then, $\forall P_0 \geq 0$, the iteration $V_{k+1} = f_{p_1p_2\ldots p_r}(V_k)$ is bounded.

Proof. The proof is similar to the proof of Lemma 3.2. □

The condition of $\sum_{i=1}^{r} p_i < 1$ may not be satisfied for most systems. Hence, another alternative condition is given.

With the same total measurement information, we suppose $p_1 \geq p_2 \geq \cdots \geq p_r$ and analogously define

$$\begin{aligned} h_{p_1p_2\ldots p_r}(X) = AXA' + Q - \sum_{i=1}^{r-1}(p_i - p_{i+1})\times \\ AX\mathcal{C}'_{2^i-1}(\mathcal{C}_{2^i-1}X\mathcal{C}'_{2^i-1} + \mathcal{R}_{2^i-1})^{-1}\mathcal{C}_{2^i-1}XA' \\ + p_rAX\mathcal{C}'_N(\mathcal{C}_NX\mathcal{C}'_N + \mathcal{R}_N)^{-1}\mathcal{C}_NXA' \end{aligned}$$

Similarly we derive the stability condition for $P_{k+1} = h_{p_1p_2\ldots p_r}(P_k)$ in the following theorem without proof.

Theorem 2. *Suppose (A, Q) is controllable and $p_1 \geq p_2 \geq \cdots \geq p_r$. If $\exists \bar{Z}_1, \bar{Z}_2, \ldots, \bar{Z}_r$ and $0 < \bar{Y} \leq I$ such that $\Psi_r^2(Y, Z_1, Z_2, \ldots, Z_r) =$*

$$\begin{bmatrix} Y & Y_r & Y_{r-1} & \ldots & Y_1 & Y_0 \\ * & Y & 0 & \ldots & 0 & 0 \\ * & * & Y & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ * & * & * & \ldots & Y & 0 \\ * & * & * & \ldots & * & Y \end{bmatrix} > 0, \qquad (24)$$

where $Y_i = \sqrt{p_i - p_{i+1}}(YA + Z_i\mathcal{C}_{2^i-1})$, $i = 1, 2, \ldots, r-1$, $Y_r = \sqrt{p_r}(YA + Z_r\mathcal{C}_N)$ and $Y_0 = \sqrt{1 - p_1}YA$.

Then, $\forall P_0 \geq 0$, the iteration $V_{k+1} = h_{p_1p_2\ldots p_r}(V_k)$ is bounded.

Remark 4. *To use the above LMI condition, we need to know which partial observation should be assigned more delivery rate a priori. The basic intuition is to refer to the eigenvalue of the state matrix, for the larger eigenvalue, the dynamic of the corresponding state is faster and, thus, the delivery rate of the corresponding partial observation should set bigger. We can also use the intuition in Remark 3 to refer to the critical rates. By test the LMI feasibility with one delivery rate decreasing and the other rates fixed on 1, the critical rates for all packets can be abtained. And rank the supposed rates according to the their computed critical rates. $p_1 \geq p_2 \geq \cdots \geq p_r$ can always be satisfied by adjusting the positions of C_i in the measurement matrix C, see Example 3.*

Similar to Lemma 3.3, we can also get the solution of $\bar{V}_h = h_{p_1p_2\ldots p_r}(\bar{V}_h)$, the upper bound of iteration $P_{k+1} = h_{p_1p_2\ldots p_r}(P_k)$.

Table 2: Critical rates derived by diffrent LMI conditions

Example	LMI	Critical delivery rates		
		p_1	p_2	p_3
2	(17)	0.32760	0.30556	0.17400
	(23)	0.32754	0.30556	0.17400
	(24)	0.32814	0.30556	0.17356
3	(17)	0.69200	0.17297	0.35885
	(24)	0.69200	0.17335	0.35885

4 NUMERIACL SIMULATION

In this section, we provide two numerical examples to illustrate the above results. All the simulations are operated on MATLAB 2014. The equation (17)(23) and (24) are programed with LMI toolbox and the semidefinite programming problem (18) is solved with Yalmip toolbox. For convenience of comparing, we sum up the outcomes in the Tab. 2.

Example 2. *Consider* $A = \begin{bmatrix} 1.22 & 0 & 0 \\ 1 & 1.1 & 0 \\ 0 & 0 & 1.2 \end{bmatrix}$, $C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. *According to the eigenvalues of A, we rank $p_1 \geq p_2 \geq p_3$. By testing the LMI conditions of* (17)(23) *and* (24)*, we derive the critical rates* $(0.32760, 0.30556, 0.17400)$ $(0.32754, 0.30556, 0.17400)$ *and* $(0.32814, 0.30556, 0.17356)$ *for* (p_1, p_2, p_3) *respectively. The p_2 are all the same because the observed state is decoupled with the other states.*

Example 3. *Consider* $A = \begin{bmatrix} 1.25 & 0 & 0 \\ 1 & 1.1 & 0 \\ 0 & 1 & 1.8 \end{bmatrix}$, $C = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $Q = 20I_3$ *and* $R = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} = 2.5I_3$. *Where I_3 is 3×3 identity matrix. By testing* (17)*, we get two critical rates* $(0.69179, 0.46999, 0)$ *and* $(0.69200, 0.17297, 0.35885)$. *Rank $p_1 \geq p_2 \geq p_3$ and test* (24)*, the critical rates are* $(0.69200, 0.46998, 0)$. *If we rank $p_1 \geq p_3 \geq p_2$, we need to revise the measurement matrix as $C = [C_1'C_3'C_2']'$. So we have $\mathcal{C}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$, $\mathcal{C}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\mathcal{C}_5 = C$ for LMI* (24)*. Then another critical rates* $(0.69200, 0.17335, 0.35885)$ *is obtained . Since* $(A, \begin{bmatrix} C_1 \\ C_2 \end{bmatrix})$ *is observable, as long as $p_1 \geq 0.692$ and $p_2 \geq 0.470$, the $\lim_{k\to\infty} \mathbb{E}[P_k]$ is bounded for any rate p_3 in $[0, 1]$. The trace of upper bound $\bar{V}$ and $\bar{V}_h$ are plotted in Fig.2 when p_1, p_2, p_3 approach to the critical rates in a straight line.*

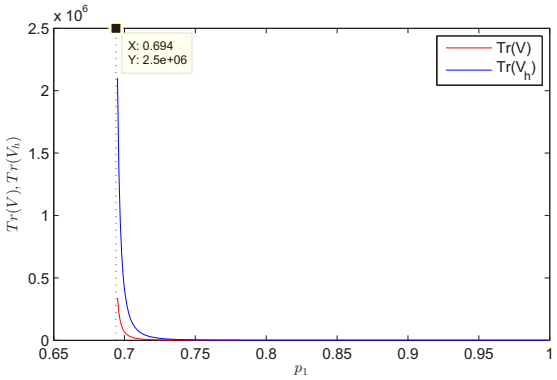

Fig. 2: Transition to instability of upper bound ($\bar{V}$) and ($\bar{V}_h$) with $\bar{V} = g_{p_1 p_2 \ldots p_r}(\bar{V})$ and $\bar{V}_h = h_{p_1 p_2 \ldots p_r}(\bar{V}_h)$ respectively when (p_1, p_2, p_3) changes from $(1, 1, 1) \longrightarrow (0.694, 0.480, 0.023)$ in a space straight line.

5 CONCLUSION

In this paper, we formulate the Kalman filtering problem with randomly received packets. We inllustrate the random sequence of the received packets in the queue have no effect on the iteration of the time-varying Kalman filter. Similar results originally done by Sinopoli etc. are present in multi-packet transmission case. Two alternative condition which, to a great extent, can work as the original condition, are also present. By iterating the LMI feasibility problem, we can get some critical rates for each packets. The simulation results have shown that the critical rates derived by the alternative condition is close to the outcomes derived by the original condition in high accuracy.

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